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Some Navigation Rules for D-Brane Monodromy

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Abstract

We explore some aspects of monodromies of D-branes in the Kähler moduli space of Calabi–Yau compactifications. Here a D-brane is viewed as an object of the derived category of coherent sheaves. We compute all the interesting monodromies in some nontrivial examples and link our work to recent results and conjectures concerning helices and mutations. We note some particular properties of the 0-brane.

1 Introduction

There has been something of an evolution in our ideas about how a D-brane should be considered. For the purposes of this paper we are interested only in the even-dimensional branes in a type II string (so-called "B-branes"). The sequence of a ideas have progressed roughly as follows

1. A D-brane is something on which an open string may end.
2. A D-brane is a $\mathrm{U}(N)$ gauge theory living on a subspace with scalar fields spanning the normal bundle.
3. A D-brane should be viewed as coming from K-theory [1, 2].
4. A D-brane should be viewed as an object of the derived category[1] of coherent sheaves [3, 4] (see also [5]).

We could also add that for non trivial H field a D-brane should be viewed as an object of the derived category of sheaves of modules over an Azumaya algebra [6]. We will assume H is trivial and so view D-Branes as an object of the derived category of coherent sheaves. We will consider our target space to be a Calabi–Yau threefold X and we denote the derived category in question as $\mathbf{D}(X)$. For the purposes of this paper we ignore any issues concerning the stability of D-branes. Our D-branes, which are objects of $\mathbf{D}(X)$, were called "topological D-branes" in [4] where issues of stability were discussed.

The reason that $\mathbf{D}(X)$ is "better" than K-theory is that it contains so much more information. For example *any* 0-brane on X corresponds to the same single element of K-theory whereas the object of $\mathbf{D}(X)$ corresponding to a 0-brane knows *where* this point is. That is, K-theory measures the charge of the D-brane but $\mathbf{D}(X)$ tells us more and possibly all we could wish to know about a particular D-brane.

As well as being knowledgeable about D-branes, $\mathbf{D}(X)$ is also very knowledgeable about X itself. This shouldn't be too surprising as if we know about all the 0-branes on X then we know about all the points on X and so we should know about X itself. Indeed for a very large class of algebraic varieties it was shown by Bondal and Orlov [7] that X is completely determined, as an algebraic variety, by $\mathbf{D}(X)$. While this process doesn't quite work for Calabi–Yau varieties it is easy to speculate that adding such data as the spectrum of central charges while in a "Calabi–Yau phase" may provide the missing information. This would allow the target space to be constructed only given worldsheet information. Clearly therefore the derived category should be of great *physical* as well as mathematical interest.

An interesting question, also studied in [7], concerns in how many ways one may associate a derived category to a fixed X. An "autoequivalence" of a derived category is a map from

[1] In this paper the derived category will always be bounded at both ends.

the category to itself preserving all the intrinsic algebraic structure associated to the category. Such a map need not preserve D-branes themselves. For example an object representing a 2-brane may become something which more resembles a 4-brane under such a transformation. In terms of string theory, these autoequivalences can arise from monodromy in the moduli space of the complexified Kähler form as first observed by Kontsevich [3]. Indeed, the fact that this monodromy action on $\mathbf{D}(X)$ can be understood at all is one the appealing aspects of the derived category. It was suggested in [8] that the derived category should play a rôle in the heterotic string for similar reasons.

Since many interesting questions about $\mathbf{D}(X)$ are associated to these monodromies, the purpose of this paper is to explore some of the aspects of these monodromies. The analysis of monodromies when the moduli space of complexified Kähler forms has only one complex dimension is pretty easy as we review in section 3. Most of the interesting properties of monodromy do not appear until we explore higher-dimensional moduli spaces. We do this in the later sections.

One should note that many of these problems can be, and have been, performed using the method of solving the Picard–Fuchs equations and using analytic continuation (see [9–13] etc.). These methods have been used in the context of D-branes in such papers as [14–17]. Instead we will use the language of derived categories where, we believe, the structure is much simpler to understand. In this way, computation of the monodromy is extremely easy (at least on the cohomology classes) and does not require the aid of a computer. Note that we are not being at all original in using the derived category — computations along these lines have been done in [18–21] for example. Our work differs from the latter only in the way we probe more deeply into the moduli space addressing such questions as monodromy around Landau–Ginzburg points in multi-parameter examples. Monodromy using the derived category approach has also been studied recently in papers such as [22].

Some interesting papers [23–25] have appeared recently which compute the finite monodromy associated to orbifold theories by using the method of "helices and mutations of exceptional sheaves". One of the motivations of this paper was to better understand this construction in the language of the derived category. We discuss the connection (and differences) in section 4. We also study the case of reducible exceptional divisors in section 6.2 which appears, at least at first sight, to lie somewhat outside the method of helices.

At least in the context of Batyrev-type Calabi–Yau varieties associated to toric geometry [26], it seems possible to rigorously classify all types of monodromy. Indeed Horja [20] has achieved this in the neighbourhood of the large radius limit. Rather than attempt such a classification we will simply go through some examples which appear to demonstrate most of the interesting things which can happen.

Our analysis is closely tied to the "phase picture" [27, 28] of the moduli space. One has various limit points in the moduli space each of which lies in the centre of some phase. There are naturally embedded $\mathbb{P}^1$'s in the moduli space which connect adjacent limit points. We are concerned with monodromy within, or almost within, such $\mathbb{P}^1$'s. Most of the "interesting"

questions one could ask about monodromy appear to be contained in this structure.

In section 5 we exhaustively study a two-parameter example obtaining all interesting monodromies associated to this model. In section 6 we study some aspects of another couple of examples which exhibit some properties not seen in section 5.

Because of the prominent rôle played by the 0-brane in the construction of Bondal and Orlov, we discuss some of its properties under monodromies in section 7.

2 Autoequivalences and the Fourier-Mukai Transform

In this section we will quickly review the language we use for the derived category. See [29,30] for more information about the derived category itself and [19,31] for more on some of the notation used below.

Douglas [4] has argued that the even-dimensional D-branes on a Calabi–Yau X should be associated with objects in the derived category of coherent sheaves on X. The morphisms in this category are associated to open strings. Given a particular object $\mathscr{K}$ of $\mathbf{D}(X \times X)$ we associate projections

$$\begin{tikzcd} & X \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd} \tag{1}$$

and the Fourier–Mukai transform [32,33]

$$T_{\mathscr{K}}(\mathscr{F}) = \mathbf{R}p_{2*}(\mathscr{K} \overset{\mathbf{L}}{\otimes} p_1^* \mathscr{F}), \tag{2}$$

for any object $\mathscr{F}$ of $\mathbf{D}(X)$.

If $\mathscr{K}$ is chosen carefully (see [7,34] for details) then the Fourier–Mukai transform will be an "autoequivalence" of $\mathbf{D}(X)$. Namely it maps $\mathbf{D}(X)$ back to itself while preserving the important algebraic structure associated to the "distinguished triangles". What this means for us is that the physics should remain invariant under such a transformation.

There are two cases of such $\mathscr{K}$'s which are of particular interest which will be denoted $\mathscr{K}^B$ and $\mathscr{K}^K$. First let L be a line bundle (or invertible sheaf) over X and let $j : X \to X \times X$ be the diagonal embedding. Then let $\mathscr{K}^B_L$ (where the superscript B stands for "B-field" for reasons to become clear) be the object of $\mathbf{D}(X \times X)$ given by

$$\ldots \to 0 \to j_* L \to 0 \to \ldots\,, \tag{3}$$

such that the nontrivial term is at the 0th position.

Now consider the object of $\mathbf{D}(X)$ given by a sheaf at 0th position:

$$\ldots \to 0 \to \mathcal{F} \to 0 \to \ldots \tag{4}$$

If we apply to (4) the Fourier–Mukai transform associated to $\mathscr{K}_L^B$, we obtain

$$\ldots \to 0 \to \mathcal{F} \otimes L \to 0 \to \ldots \tag{5}$$

To relate this to string theory, let $\mathcal{F}$ be a sheaf supported over some subspace of X. That is, we have a D-brane wrapping this subspace. All we have done by applying this transform is to change the field-strength of the U(1) gauge bundle over the D-brane. Gauge invariance forces the combination $F - B$ to be appear in the action of the D-brane. The above transformation must therefore be equivalent to $B \mapsto B + L$. (Note that we use L to denote the line bundle, the associated divisor class and the dual 2-form $c_1(L)$.) Thus we may assert (as was also done in [20]) that *the transformations associated to $\mathscr{K}_L^B$ are those of a B-field shift $B \mapsto B + L$.*

Another transformation of interest is that of Seidel and Thomas [19] given by

$$\mathscr{K}_{\mathscr{E}}^K = \mathrm{Cone}\Big\{\mathscr{E}^\vee \boxtimes \mathscr{E} \to j_*\mathcal{O}_X\Big\}, \tag{6}$$

where $\mathcal{O}_X$ is the structure sheaf of X. The superscript K stands for "Kontsevich" who was the first to use this kind of transformation in the context of string theory [18]. Here the notation $A \boxtimes B$ is short for $p_1^*A \otimes p_2^*B$. The object $\mathscr{E}$ is any object of $\mathbf{D}(X)$ which satisfies the sphericity conditions given in [19]. We refer to [30] for a nice description of the cone construction.

Now the associated Fourier–Mukai transform simplifies to the following [19]:

$$T_{\mathscr{K}_{\mathscr{E}}^K}(\mathscr{F}) = \mathrm{Cone}\Big\{\mathrm{Hom}(\mathscr{E},\mathscr{F}) \otimes \mathscr{E} \xrightarrow{f} \mathscr{F}\Big\}, \tag{7}$$

where f is the obvious evaluation map.

It is worth pointing out a subtle but potentially important point. The cone construction is not a particularly well-defined functor in the context of the derived category. We refer to section 1.4 of chapter 5 of [29] for a discussion of the problems. By writing the transformation in the form (7) we potentially expose ourselves to such ambiguities. One should always bare in mind however that the transformation exists as a Fourier–Mukai transform (2) which yields a perfectly well defined functor from $\mathbf{D}(X)$ to itself. In particular the cone appearing in (6) is only defining $\mathscr{K}_{\mathscr{E}}^K$ as an object and no functorial properties of the cone are required there.

It is difficult to understand the physical meaning of the transformation associated to $\mathscr{K}_{\mathscr{E}}^K$ working directly in the derived category. Instead we take Chern characters to see the effect on cohomology. From (7) one deduces that [19]

$$ch(T_{\mathscr{K}_{\mathscr{E}}^K}(\mathscr{F})) = ch(\mathscr{F}) - \langle \mathscr{E},\mathscr{F}\rangle \, ch(\mathscr{E}), \tag{8}$$

where

$$\begin{aligned}\langle \mathscr{E}, \mathscr{F} \rangle &= \sum_i (-1)^i \dim \operatorname{Hom}^i(\mathscr{E}, \mathscr{F}) \\ &= \int_X ch(\mathscr{E}^\vee)\, ch(\mathscr{F})\, td(\mathscr{T}_X),\end{aligned} \tag{9}$$

and $\mathscr{T}_X$ is the tangent sheaf of X.

Now it is generally believed that the (skew-symmetric) inner product $\langle \mathscr{E}, \mathscr{F} \rangle$ on X is equal to the (equally skew-symmetric) intersection form for 3-cycles on the mirror Y [14, 35, 36]. According to this analogy, the transformation (8) is nothing more than a Picard–Lefschetz transformation that one would associate to monodromy around a vanishing 3-sphere in Y [19, 36]. Because of this it seems natural to expect this kind of transformation to be associated to monodromy in the moduli space of complex structures around some parts of the discriminant locus.

3 A One Parameter Case

In this section we will review a computation apparently first done by Kontsevich [18].

We consider the case where X is the quintic hypersurface in $\mathbb{P}^4$. As is very well-known [9] the moduli space of complexified Kähler forms can be taken to be $\mathbb{P}^1$ with three interesting point. These point are as follows:

P_0: The Gepner point. Metrically it lies at an orbifold point $\mathbb{C}/\mathbb{Z}_5$.

P_1: The "conifold point". The mirror of X acquires a conifold singularity. The conformal field theory associated to X is singular.

P_∞: The large radius limit. This point is an infinite distance away from the above two points.

Let H denote the homology class of the 4-cycle given by the hyperplane section of the quintic 3-fold. We will also use H to denote its Poincaré dual which generates $H^2(X, \mathbb{Z})$. We then have

$$\begin{aligned} td(\mathscr{T}_X) &= \frac{\left(\dfrac{H}{1-e^{-H}}\right)^5}{\left(\dfrac{5H}{1-e^{-5H}}\right)} \\ &= 1 + \tfrac{5}{6}H^2, \end{aligned} \tag{10}$$

and

$$\int_X H^3 = 5. \tag{11}$$

The monodromy around P_0 is expected to be of order 5 because of the $\mathbb{Z}_5$ quantum symmetry of the Gepner model. The monodromy around P_∞ is known to correspond to $B \mapsto B + H$. In other words we expect it to be given by the transformation $\mathscr{K}^B_{\mathcal{O}(H)}$. We will denote this by $\mathscr{K}^B_H$ for short.

Kontsevich conjectured that the monodromy around P_1 is given by the Fourier–Mukai transform $\mathscr{K}^K_{\mathscr{E}}$ of the previous section where $\mathscr{E}$ is given by the structure sheaf $\mathcal{O}_X$. We denote this $\mathscr{K}^K_0$ for short. We will return to this conjecture in a more precise form in section 5.

It follows from the topology of a sphere with 3 punctures that the product[2] of the monodromy around P_∞ and the monodromy around P_1 should equal the monodromy around P_0 and hence should be of order 5. We may verify that this is consistent with the Chern character of any starting D-brane.

Let us start with $\mathscr{F}$ given by $\mathcal{O}_X$ and apply the desired sequence of monodromy transformations. From (8) we obtain the following:

$$\begin{aligned} ch(\mathscr{F}) &= 1 \\ ch(\mathscr{K}^B_H\mathscr{F}) &= e^H \\ ch(\mathscr{K}^K_0\mathscr{K}^B_H\mathscr{F}) &= e^H - 5 \\ ch(\mathscr{K}^B_H\mathscr{K}^K_0\mathscr{K}^B_H\mathscr{F}) &= e^{2H} - 5e^H \\ &\vdots \\ ch((\mathscr{K}^K_0\mathscr{K}^B_H)^5\mathscr{F}) &= 1 + (e^H-1)^5 = 1, \end{aligned} \tag{12}$$

as $H^4 = 0$ in X. This is therefore consistent.

It would be interesting to check that $(\mathscr{K}^K_0\mathscr{K}^B_H)^5$ gives the identity transform when applied directly to $\mathbf{D}(X)$ rather than just applied to cohomology. We will not attempt this here.

Note that it is easy to apply the same method to other one parameter examples as listed in [37] for example.

[2]Note that we are required to take the loops around P_1 and P_∞ in the "same direction" in order for their product to be a loop around P_0. Throughout this paper we will have orientation problems such as this. We will not concern ourselves at all with such details. In all the examples we do, we simply find the right combination which gives the expected results.

4 Helices and Mutations

The purpose of this section is to briefly point out similarities and differences between the above computation for the quintic and the notion of helices and exceptional sheaves. The reader who is not directly interested in such things can skip this section.

It is easiest to describe mutations and helices directly in the derived category. See [38] for example for more information. Let us consider the space $V = \mathbb{P}^4$ and let H denote the hyperplane class. Now consider the exceptional collection of sheaves $\{\mathcal{O}, \mathcal{O}(H)\}$. We may use a mutation to pull $\mathcal{O}(H)$ to the left through $\mathcal{O}$. Let $\mathscr{E}(H)$ denote the resulting object in $\mathbf{D}(\mathbb{P}^4)$. One can compute $ch(\mathscr{E}(H)) = e^H - 5$.

Similarly we may begin with the set $\{\mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H)\}$ and pull $\mathcal{O}(2H)$ through $\mathcal{O}(H)$ and $\mathcal{O}$ to obtain $\mathscr{E}(2H)$ etc. The result of such mutations appears remarkably similar to the monodromy transformations we considered in the previous section. Indeed one obtains

$$ch(\mathscr{E}(nH)) = ch((\mathscr{K}_0^K \mathscr{K}_H^B)^n \mathscr{F}), \quad \text{for } n = 0, \ldots, 4. \tag{13}$$

This correspondence fails for $n = 5$ however. In this case we have $ch(\mathscr{E}(5H)) = 0$. This disagreement should come as no surprise. The language of mutations of helices can be recast in the form of the Fourier–Mukai transforms of section 2. The key point however is that the algebraic variety in question is the ambient $\mathbb{P}^4$ itself rather than the Calabi–Yau hypersurface X. Indeed exceptional sheaves cannot exist on X.

A Fourier–Mukai transformation does *not* yield an autoequivalence of $\mathbf{D}(\mathbb{P}^4)$ as it fails the canonical class constraint of [34]. That is why the fifth application of the supposed monodromy transformation can kill the object in $\mathbf{D}(\mathbb{P}^4)$.

The language of mutations of helices was used in [23–25] successfully to obtain monodromies because equation (13) holds true. Note however that the procedure of going from $\mathscr{E}(nH)$ to $\mathscr{E}((n+1)H)$ cannot generally be identified with monodromy around the Gepner point because of the more general failure of this relation.

5 A Two Parameter Case

The structure of monodromies becomes considerably more interesting when one starts to look at moduli spaces of more than one dimension. In this case, the "discriminant locus" of bad conformal field theories is a subvariety of the moduli space with dimension one or more.

We wish to see if the Fourier–Mukai transforms considered above can also be applied in these more complicated situations. Note that this problem has been studied by Horja [20] and by Seidel and Thomas [19]. Horja gave extensive results for monodromy loops which are "close" to the large radius limit in some sense. We will be interested in relations obtained by venturing further into the moduli space. The Fourier–Mukai transforms we will consider follow closely the construction by Seidel and Thomas [19]. The paper [39] has appeared very recently which shows that these methods are essentially a special case of Horja's construction.

Our goal in this section will be to obtain a similar result to section 3 in a two parameter example. Namely, can we find a sequence of monodromies which give a loop around some point which looks like a Gepner point, and hence has finite order?

Because the moduli space is two-dimensional there is no notion of "monodromy around a point". Given a complex curve in our moduli space we can define a monodromy. We will use two notions extensively in the text below and we wish to emphasize the difference here to avoid confusion. We will often refer to monodromy *around* a curve for a loop in the two-dimensional moduli space which lies external to the curve. We will also refer to to the completely different notion of monodromy *within* the curve around a specific point. The words "around" and "within" will always have the above meaning.

Our example is where X is a blown-up hypersurface of degree 18 in the weighted projected space $\mathbb{P}^4_{9,6,1,1,1}$. This space was studied extensively in [12] and analyzed in relation to D-branes in [17]. The two generators of $H^2(X,\mathbb{Z})$ (and their Poincaré dual divisors) will be called H and L consistent with [12]. If we put homogeneous coordinates $[z_1,\dots,z_5]$ on $\mathbb{P}^4_{9,6,1,1,1}$ then the divisor class of $z_1=0$ is given by $3H$, the class of $z_2=0$ is given by $2H$. The class of $z_3=0$ or $z_4=0$ or $z_5=0$ (after the blow-up) is given by L. This weighted projective space has a curve of singularities which intersects the hypersurface at one point. Locally this point looks like the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. We blow this up to produce an exceptional divisor $E\cong\mathbb{P}^2$. In terms of homology classes, $E=H-3L$.

The remaining topological information for X required is as follows. H^2, HL, L^2 live in $H_2(X)$ or $H^4(X)$ subject to the constraint $H(H-3L)=0$. $H_0(X)$ or $H^6(X)$ has a single generator we denote p, and $H^3=9p$, $H^2L=3p$, $HL^2=p$, $L^3=0$. Obviously any monomial of degree 4 or higher in H or L vanishes. Finally

$$td(\mathscr{T}_X)=1-\tfrac{1}{2}L^2+\tfrac{1}{4}HL. \tag{14}$$

The mirror, Y, of X has defining equation

$$a_0z_1z_2z_3z_4z_5+a_1z_1^2+a_2z_2^3+a_3z_3^{18}+a_4z_4^{18}+a_5z_5^{18}+a_6z_3^6z_4^6z_5^6. \tag{15}$$

The "algebraic" coordinates on the moduli space are then given by

$$x=\frac{a_1^3a_2^2a_6}{a_0^6},\quad y=\frac{a_3a_4a_5}{a_6^3}. \tag{16}$$

We may then define the discriminant as an expression in x and y which vanishes when Y becomes singular. If the data can be presented torically as in this case then section 3.5 of [40] gives a nice efficient way of computing this discriminant. In our example, the discriminant factorizes into two parts:

$$\begin{aligned}\Delta_0&=6^{12}x^3y+(432x-1)^3\\ \Delta_1&=27y+1.\end{aligned} \tag{17}$$

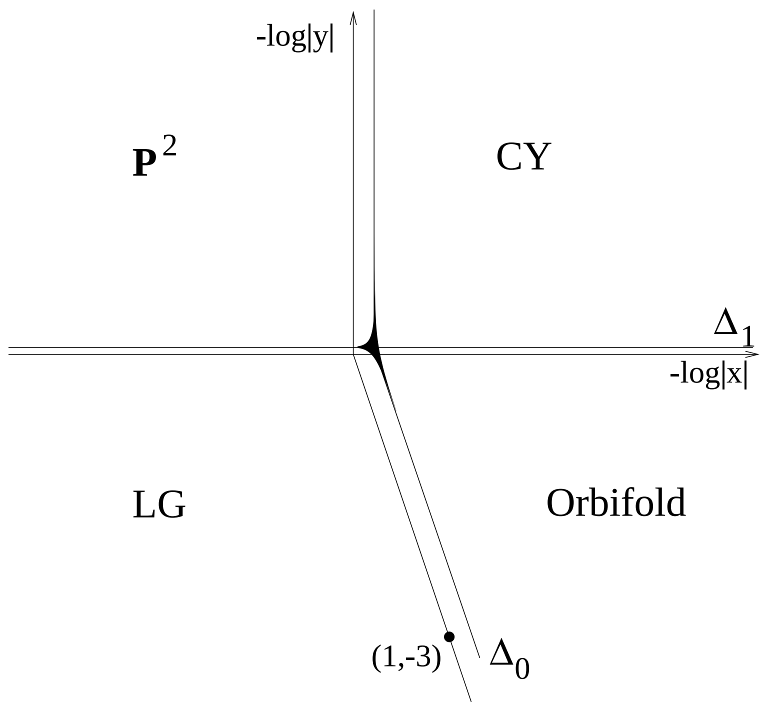

Figure 1: The 4 phases of the 2 parameter example.

We want to picture the moduli space in two different ways. First we use the "phase" description of [27, 28]. See also section 3 of [41]. We project the discriminant into $\mathbb{R}^2$ by plotting $-\log|y|$ against $-\log|x|$. We show the result in figure 1. The result is that the 2-plane is divided into four "phases". We chose our algebraic coordinates (16) so that the Calabi–Yau phase appears in the positive quadrant. The limit point of this phase is the large radius limit. The positive quadrant may also be viewed as the Kähler cone of X where the class H gives the horizontal direction and L gives the vertical direction.

The other phases are pictured as follows. There is an orbifold phase whose limit point has the orbifold singularity $\mathbb{C}^3/\mathbb{Z}_3$ but the Calabi–Yau has infinite volume. There is a $\mathbb{P}^2$ phase where X collapses onto a $\mathbb{P}^2$. This can happen as X is an elliptic fibration over $\mathbb{P}^2$. In the limit, this elliptic fibre has zero area and the $\mathbb{P}^2$ has infinite volume. Finally we have a Landau–Ginzburg phase with the Gepner point as the limit point.

The other way of drawing the moduli space is as a complex surface. The phase picture in figure 1, i.e., the *secondary fan* of X, is viewed as the fan of a toric variety $\mathscr{M}$ as described in [28]. The coordinates x and y are then naturally coordinates over a patch of the moduli space $\mathscr{M}$.[3] In this case $\mathscr{M}$ is a surface with two quotient singularities. The discriminant is a divisor in $\mathscr{M}$.

[3]This is identified by associating the dual of the cone in the positive quadrant with $\operatorname{Spec}\mathbb{C}[x,y]$.

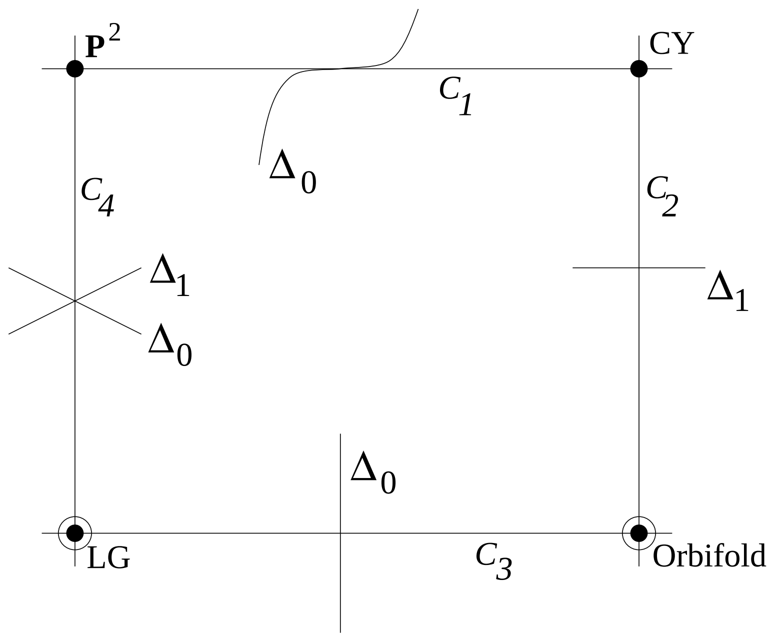

Figure 2: The moduli space $\mathscr{M}$.

We sketch $\mathscr{M}$ in figure 2 by drawing complex dimensions as real. Our limit points appear as dots in the diagram. We draw the curves $C_1, \ldots, C_4$ as the $\mathbb{P}^1$'s joining adjacent phase limits. Torically these curves are associated to the lines in figure 1 which separate the phases. Figure 2 also shows how the discriminant intersects these curves. Note that Δ_0 and Δ_1 are themselves smooth curves in $\mathscr{M}$. The extra circles around the LG and orbifold point denote the fact that both of these points lie at a quotient singularity of the form $\mathbb{C}^2/\mathbb{Z}_3$ in $\mathscr{M}$.

If we followed the analysis of [12] we would now blow-up $\mathscr{M}$ so that it was smooth and that the intersections of the discriminant with the curves C_i were transverse. This requires several blow-ups. Rather than do this, we find it easier to work directly in $\mathscr{M}$.

We would now like to take each of the curves $C_1, \ldots, C_4$ in turn and do a similar computation to that of section 3 *within* that curve (or nearly so) to check our monodromy predictions. Note that each curve C_i has three special points on it — the two limit points and a third point where the discriminant hits the curve in some way. Thus, just as in section 3, we will show that the product of the monodromy around one of the limit points and around the discriminant is equal to the monodromy around the other limit point.

5.1 C_1

Let us fix a basepoint near the large Calabi–Yau limit point. Because we have identified the cone of this phase with the Kähler cone of X we immediately know the monodromies around C_1 and C_2. Each must be a shift in the B-field. To be precise, a loop around C_1

will correspond to $B \mapsto B + L$ and hence corresponds to $\mathscr{K}_L^B$. Similarly a loop around C_2 is given by $\mathscr{K}_H^B$.

Since C_1 and C_2 intersect transversely, the monodromy around the Calabi–Yau point within C_1 corresponds to going around C_2 and is thus given by $\mathscr{K}_H^B$. In other words $ch(\mathscr{F}) \mapsto e^H\, ch(\mathscr{F})$.

What about the monodromy within C_1 around the point in the middle where the discriminant hits? A method for computing this was presented in [20] but we will proceed a little differently. First we need to decide how to go around a generic part of Δ_0. Note that Δ_0 is the irreducible component of the discriminant corresponding to the appearance of singularities in (15) for *nonzero* $z_1, \ldots, z_5$. This was dubbed the "A-discriminant" in [42].[4] We will call it the "*primary*" component of the discriminant. One could also define this as the component which is computed by finding solutions to equation (3.45) of [40]. We then state the following conjecture which appears to be due to Horja, Kontsevich and Morrison in some form or another [18, 20, 43].

Conjecture 1 *For a suitable choice of basepoint near the Calabi–Yau limit point, a loop around the primary component of the discriminant is given by* $\mathscr{K}_0{}^K = \mathscr{K}_{\mathscr{E}}^K$, *with* $\mathscr{E} = \mathcal{O}_X$.

This is certainly consistent with section 3 where the primary component was the entire discriminant.

Assuming this conjecture to be true we still have a complication that makes the computation a little less straight-forward. Namely, C_1 does not intersect Δ_0 transversely but rather intersects it tangentially with multiplicity 3. This means that we cannot say that the monodromy within C_1 around the discriminant is given by the above conjecture.

To proceed with the computation we can put a small 3-sphere S^3 around the intersection of Δ_0 and C_1. Since Δ_0 and C_1 are both smooth it follows that $L_1 = S^3 \cap C_1$ and $L_2 = S^3 \cap \Delta_0$ are both unknotted circles. Because of the tangential intersection, these circles are linked three times. We may remove a point of S^3 and imagine the link in $\mathbb{R}^3$. We show this in figure 3.

Next we need to describe $\pi_1(S^3 - (L_1 \cup L_2))$. To do this we use the Wirtinger presentation (see [44] for example). Imagine fixing a basepoint above the sheet of paper. The arrow a in figure 3 then represents the element of π_1 looping under the left circle in the direction indicated by the arrow. Similarly we define b for the right circle. We then have further

[4] Perhaps rather confusingly, [42] uses the term "principal A-determinant" for the full discriminant $\Delta_0\Delta_1$. Even more confusingly, Δ_0 has sometimes been called the "principal component" of the discriminant [20].

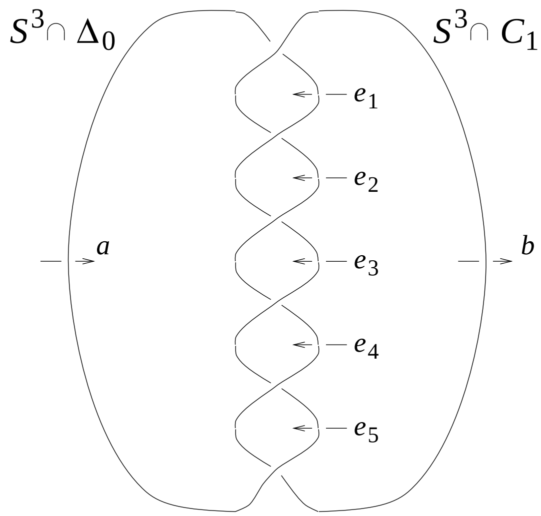

Figure 3: The triply linked circles.

elements e_i as shown in the figure. Crossing relations then determine[5]

$$\begin{aligned}
e_1 &= b^{-1}ab \\
e_2 &= b^{-1}a^{-1}bab \\
e_3 &= b^{-1}a^{-1}b^{-1}abab \\
e_4 &= b^{-1}a^{-1}b^{-1}a^{-1}babab \\
e_5 &= b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}ababab,
\end{aligned} \tag{18}$$

but clearly $e_5 = a$ which yields the relation

$$ababab = bababa. \tag{19}$$

Indeed $\pi_1(S^3 - (L_1 \cup L_2))$ is given by the group on two generators (a, b) subject to the single relation (19).

Deform the path within C_1 around the discriminant point a little so that it lies outside C_1. This is the path around which we wish to compute the monodromy. This is the (clockwise) path which follows closely the circle $S^3 \cap C_1$. Such a path is homotopic to $e_1 e_3 e_5 = b^{-2}ababa$. But a is nothing more than a generic loop around Δ_0 and so is given by $\mathscr{K}_0^K$, and similarly b is given by $\mathscr{K}_L^B$. Therefore we claim that monodromy around the discriminant point within C_1 is given by $(\mathscr{K}_L^B)^{-2}\mathscr{K}_0^K\mathscr{K}_L^B\mathscr{K}_0^K\mathscr{K}_L^B\mathscr{K}_0^K$.

Let Q_1 denote the monodromy within $\mathbb{C}_1$ around the $\mathbb{P}^2$ limit point. As this is equal to the combined monodromy around the discriminant point and the Calabi–Yau limit point, we have $Q_1 = \mathscr{K}_H^B(\mathscr{K}_L^B)^{-2}\mathscr{K}_0^K\mathscr{K}_L^B\mathscr{K}_0^K\mathscr{K}_L^B\mathscr{K}_0^K$.

[5]Our order convention is that ab represents the path b followed by the path a.

Passing from the Calabi–Yau limit point to the $\mathbb{P}^2$ limit point represents collapsing a large radius elliptic fibre to a Landau–Ginzburg orbifold theory in a manner very similar to the collapse of the quintic in section 3. This Landau-Ginzburg theory is a $\mathbb{Z}_6$-orbifold and thus has a $\mathbb{Z}_6$ quantum symmetry. It should therefore follow that $Q_1^6 = 1$ in complete analogy with the Landau–Ginzburg point in section 3. This is a highly nontrivial check of our picture:

$$\begin{aligned}
ch(\mathcal{O}_X) &= 1 \\
ch(Q_1\mathcal{O}_X) &= e^H - 3e^{H-L} + 3e^{H-2L} \\
ch(Q_1^2\mathcal{O}_X) &= e^{2H} - 3e^{2H-L} + 3e^{2H-2L} - e^H \\
&\vdots \\
ch(Q_1^6\mathcal{O}_X) &= e^{6H} - 3e^{6H-L} + 3e^{6H-2L} - e^{5H} - e^{4H} + 3e^{4H-L} - 3e^{4H-2L} + 3e^{3H-L} \\
&\qquad - 3e^{3H-2L} + e^{2H} + e^H - 3e^{H-L} + 3e^{H-2L} \\
&= 1.
\end{aligned} \tag{20}$$

5.2 C_2

So far we have only used $\mathscr{E} = \mathcal{O}_X$ in the Fourier–Mukai transform $\mathscr{K}_{\mathscr{E}}^K$. In this section we use a less trivial choice. The curve C_2 represents the process of blowing-up up the $\mathbb{C}^3/\mathbb{Z}_3$ singularity while keeping the rest of the Calabi–Yau at infinite volume. We will attempt to "localize" the computations to around the exceptional divisor $E \cong \mathbb{P}^2$.

Let us consider the general case of an irreducible exceptional divisor E in a Calabi–Yau space X of arbitrary dimension. Let the normal bundle be denoted by N. Let us assume that the zero locus of a generic section of $N^\vee$ gives an irreducible variety $W \subset E$ of complex dimension two less than X. W is automatically Calabi–Yau. In our example, W would be a cubic curve in $\mathbb{P}^2$. We therefore have two inclusions

$$i : E \hookrightarrow X, \quad j : W \hookrightarrow E. \tag{21}$$

Now consider two objects $\mathscr{E}, \mathscr{F} \in \mathbf{D}(E)$ associated to sheaves on E. In terms of our inner product on X we may apply the Grothendieck–Riemann–Roch theorem to yield the following

localization:

$$\begin{aligned}
\langle i_*\mathscr{E}, i_*\mathscr{F}\rangle_X &= \int_X (ch(i_*\mathscr{E}))^\vee\, ch(i_*\mathscr{F})\, td(\mathscr{T}_X) \\
&= \int_E (ch(i_*\mathscr{E}))^\vee\, ch(\mathscr{F})\, td(\mathscr{T}_E) \\
&= \int_W (ch(j^*\mathscr{E}))^\vee\, ch(j^*\mathscr{F}) \frac{td(\mathscr{T}_E)}{td(N^\vee)} \\
&= \int_W (ch(j^*\mathscr{E}))^\vee\, ch(j^*\mathscr{F})\, td(\mathscr{T}_W) \\
&= \langle j^*\mathscr{E}, j^*\mathscr{F}\rangle_W.
\end{aligned} \tag{22}$$

Therefore we may compute the inner product between objects of $\mathbf{D}(X)$ which are i_* of something in $\mathbf{D}(E)$ purely in terms of the local geometry of the blow-up.

We may now apply conjecture 1 to W. In our example W is an elliptic curve and has only one deformation of complexified Kähler form. The discriminant is then a point and therefore primary with respect to W. The associated Fourier–Mukai transform for monodromy is then given by $\mathscr{K}_{\mathscr{E}}^K$ for $\mathscr{E} = \mathcal{O}_W = j^*\mathcal{O}_E$. This naturally motivates the following

Conjecture 2 *The monodromy around a component of the discriminant associated with an irreducible divisor E collapsing to a point is given by $\mathscr{K}_{\mathscr{E}}^K$ for $\mathscr{E} = i_*\mathcal{O}_E$, where i is the inclusion map.*

Note that $i_*\mathcal{O}_E$ is the structure sheaf of E extended by zero over the rest of X. It can thus be denoted by $\mathcal{O}_E$ itself.

In our example we therefore associate Δ_1 with $\mathscr{K}_{\mathscr{E}}^K$ for $\mathscr{E} = \mathcal{O}_E$ where the class of E is given by $H - 3L$. Let us use $\mathscr{K}_1{}^K$ to denote this transform. Another application of Grothendieck–Riemann–Roch quickly yields

$$ch(\mathcal{O}_E) = 1 - e^{3L-H}, \tag{23}$$

(and so $ch(\mathcal{O}_E^\vee) = 1 - e^{H-3L}$).

We are now in a position to compute all the monodromies for C_2. Around the Calabi–Yau limit we have $\mathscr{K}_L^B$. The component Δ_1 hits C_2 transversely and so the monodromy around the discriminant point is given by $\mathscr{K}_1{}^K$.

Let us use $Q_2 = \mathscr{K}_1{}^K \mathscr{K}_L^B$ to refer to monodromy around the orbifold limit point. A $\mathbb{C}^3/\mathbb{Z}_3$ orbifold has a $\mathbb{Z}_3$ quantum symmetry so one might naïvely guess that $Q_2^3 = 1$. Instead we find

$$\begin{aligned}
ch(\mathcal{O}_X) &= 1 \\
ch(Q_2\mathcal{O}_X) &= e^L \\
ch(Q_2^2\mathcal{O}_X) &= e^{2L} \\
ch(Q_2^3\mathcal{O}_X) &= e^H.
\end{aligned} \tag{24}$$

This suggests the relation $Q_2^3 = \mathscr{K}_H^B$. To see why this is so let us examine more carefully the geometry of the moduli space near the orbifold limit point. As mentioned earlier this point is actually locally the singularity $\mathbb{C}^2/\mathbb{Z}_3$. We may therefore surround this limit point, not by a sphere, but by a *lens space* $M = S^3/\mathbb{Z}_3$. Now C_2 and C_3 are both smooth curves and therefore they intersect M in unknotted circles.

Consider the free $\mathbb{Z}_3$ quotient map $q : S^3 \to M$. The intersection of C_2 and C_3 with M both lift to single circles in S^3 under q^{-1}. These circles are linked once and so π_1 of the complement of these circles in S^3 is the abelian product $\mathbb{Z} \times \mathbb{Z}$. Let G denote π_1 of the complement of C_2 and C_3 in M. Since q is a normal cover we have

$$1 \to \mathbb{Z} \times \mathbb{Z} \to G \to \mathbb{Z}_3 \to 1. \tag{25}$$

A more detailed analysis of the geometry shows that $G \cong \mathbb{Z} \times \mathbb{Z}$ where there is a particular element $g_{\mathrm{orb}} \in G$ which cannot be lifted to a loop in S^3 but such that g_{orb}^3 lifts to a loop in S^3 which loops around both C_2 and C_3.

To understand the monodromy we need to deform the loop "inside" C_2 around the orbifold point a little so that it doesn't intersect C_2 or C_3. There is no unique way to do this. The homotopy class of such a deformation is given by g_{orb} times an arbitrary number of windings *around* C_2. One might argue that the most natural lift is the reduce this extra winding around C_2 and say that the desired loop is simply given by g_{orb}.

Identifying Q_2 with g_{orb} it should then follow that the monodromy Q_2^3 is given by a loop around C_2 followed by a loop around C_3. We see that $Q_2^3 = \mathscr{K}_H^B$ is entirely consistent with this so long as *the monodromy around* C_3 *is trivial.* We also see that our natural deformation of the loop within C_2 is the correct one.

We have therefore understood the monodromy in (24) and argued that the monodromy around C_3 is trivial.

For a true localization to the orbifold point we may put $H = 0$. This has the effect of sending that component of the Kähler form of to infinity. Thus the target space looks like a resolution of $\mathbb{C}^3/\mathbb{Z}_3$. In this case the monodromy around the orbifold point really is of order three. This fact was also determined directly using the Picard–Fuchs system in [15]. Note also that any object of $\mathbf{D}(X)$ which can be written as i_* of something in $\mathbf{D}(E)$ is brought back to itself exactly after looping three times around the orbifold point.

5.3 C_3

Consider first the loop around the orbifold point within C_3. In order to understand the monodromy we need again to deform this loop a little as in section 5.2. It turns out that the simplest deformation of this loop is homotopic to the same class g_{orb} as above. Figure 2 certainly makes this statement counterintuitive! At first sight it looks like a loop within C_2 is like a loop around C_3 and a loop within C_3 looks like a loop around C_2 and these are certainly not equal. It is the quotient singularity which stops this argument working. Both

the loop within C_2 and the loop within C_3 must deform to elements of G which map to the same element of $\mathbb{Z}_3$ in (25). The most natural deformation of these two loops actually makes the loops homotopic.

Therefore monodromy around the orbifold point within C_3 is given by $Q_2 = \mathscr{K}_1^K \mathscr{K}_L^B$. The loop around the discriminant point within C_3 is easy. Since Δ_0 intersects C_3 transversely, the monodromy is given by $\mathscr{K}_0$. The monodromy around the LG point is then given by

$$Q_3 = \mathscr{K}_0^K \mathscr{K}_1^K \mathscr{K}_L^B. \tag{26}$$

What properties should we expect for Q_3? The geometry around the LG point is very similar (up to orientation questions) to the geometry around the orbifold point. In particular one may show that the loop corresponding to Q_3 is such that its third power is homotopic to a loop around C_3 and C_4. Now we know that a loop around C_3 induces no monodromy from section 5.2. The fact that C_4 intersects C_1 transversely at a smooth point in $\mathscr{M}$ tells us that the loop around C_4 is given by Q_1 from section 5.1. Thus Q_3^3 should have the same properties as Q_1. We saw in section 5.1 that Q_1 was of order 6. *It follows that Q_3 is of order 18.*

We may confirm this explicitly. E.g.:

$$\begin{aligned}
ch(\mathcal{O}_X) &= 1 \\
ch(Q_3\mathcal{O}_X) &= e^L - 3 \\
ch(Q_3^2\mathcal{O}_X) &= e^{2L} - 3e^L + 3 \\
ch(Q_3^3\mathcal{O}_X) &= e^H - 3e^{2L} + 3e^L - 1 \\
ch(Q_3^4\mathcal{O}_X) &= e^{H+L} - 3e^H + 3e^{2L} - e^L \\
ch(Q_3^5\mathcal{O}_X) &= e^{H+2L} - 3e^{H+L} + 3e^H - e^{2L} \\
ch(Q_3^6\mathcal{O}_X) &= e^{2H} - 3e^{H+2L} + 3e^{H+L} - e^H - 1 \\
ch(Q_3^7\mathcal{O}_X) &= e^{2H+L} - 3e^{2H} + 3e^{H+2L} - e^{H+L} - e^L + 3 \\
&\vdots \\
ch(Q_3^{18}\mathcal{O}_X) &= e^{6H} - 3e^{5H+2L} + 3e^{5H+L} - e^{5H} - e^{4H} + 3e^{3H+2L} - 3e^{3H+L} + 3e^{2H+2L} \\
&\quad - 3e^{2H+L} + e^{2H} + e^H - 3e^{2L} + 3e^L \\
&= 1.
\end{aligned} \tag{27}$$

Comparing closely (20) and (27) we see that Q_3^3 is not quite the same thing as Q_1 although their effect is very similar. As we have described the loops corresponding to these transformations, they actually differ by a change in basepoint and so Q_1 and Q_3^3 are only equivalent up to conjugation.

The sequence of transformations given in (27) is identical to the sequence predicted by [23–25] in the language of helices and mutations. There one begins with a sequence

of exceptional sheaves on $\mathbb{P}^4_{\{9,6,1,1,1\}}$ of the form $\{\mathcal{O}, \mathcal{O}(L), \mathcal{O}(2L), \mathcal{O}(H), \mathcal{O}(H+L), \mathcal{O}(H+2L), \mathcal{O}(2H), \ldots, \mathcal{O}(5H+2L)\}$. One then mutates all the bundles to the left to reverse their order giving a sequence of objects whose Chern characters are exactly given by (27). It is not hard to see why this is so. Roughly speaking the transformation $Q_3 = \mathscr{K}_0^K \mathscr{K}_1^K \mathscr{K}_L^B$ may be described as follows. $\mathscr{K}_L^B$ takes the sheaf $\mathcal{O}(nL)$ to $\mathcal{O}((n+1)L)$. Next $\mathscr{K}_1^K$ leaves $\mathcal{O}(mH+L)$ or $\mathcal{O}(mH+2L)$ invariant but takes $\mathcal{O}(mH+3L)$ to $\mathcal{O}((m+1)H))$. Finally $\mathscr{K}_0^K$ is a "left mutation" just as it was for the quintic in section 4.

This gives a natural explanation for the funny "jump" seen in the required sequence of exceptional sheaves from $\mathcal{O}(mH+2L)$ to $\mathcal{O}((m+1)H)$. It is effectively caused by the action of $\mathscr{K}_1^K$. Note again the following shortcoming of the method of using sheaves on $\mathbb{P}^4_{\{9,6,1,1,1\}}$. If we extend this process by adding $\mathcal{O}(6H)$ to the above sequence of sheaves then the 18th transformation of $\mathcal{O}$ would have Chern character equal to 0 rather than 1. Again this is because the corresponding Fourier–Mukai transform on $\mathbb{P}^4_{\{9,6,1,1,1\}}$ is not invertible.

5.4 C_4

Finally we do the monodromy computation within C_4. This actually yields nothing new. Let Q_4 be the monodromy given by the loop within C_4 around the LG point. Because the LG point is an orbifold point, one can show that this loop (or at least a small deformation of it) is the same as the loop within C_3 around the LG point for reasons essentially identical to the discussion in section 5.2. This immediately implies $Q_4 = Q_3$.

Indeed monodromy around the $\mathbb{P}^2$ point within C_4 is given by $\mathscr{K}_L^B$; and monodromy around the discriminant point requires loops around both Δ_1 and Δ_0 as seen in figure 2. Thus we see $Q_4 = \mathscr{K}_0^K \mathscr{K}_1^K \mathscr{K}_L^B$ consistent with the above paragraph.

Consider trying to generalize the results of this example. One can show that the specific structure of the monodromy seen in this example depends mainly on the fact that X is a fibration. For example we could consider a more complicated example with 3 Kähler moduli such as the resolved hypersurface of degree 24 in $\mathbb{P}^4_{\{1,1,2,8,12\}}$. This is an elliptic fibration over a Hirzebruch surface which itself is a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$. In this example one has a curve in the moduli space which is the analogue of C_4 above. It connects the Landau–Ginzburg phase with the $\mathbb{P}^1$ phase. Monodromy around the LG point can then be shown to be of a form $\mathscr{K}_0^K \mathscr{K}_1^K \mathscr{K}_2^K \mathscr{K}_L^B$. This will then reproduce the results of section 9.3 of [25] for example. Note however that the fibration structure is essential here.

6 Other Examples

The example of section 5 demonstrated many features of monodromy but there are many other important possibilities which did not appear. In this section we discuss some examples which do exhibit these effects.

6.1 Surfaces shrinking to curves

All the monodromies around components of the discriminant have been given by a Fourier–Mukai transform of the form (6) studied by Seidel and Thomas. Here we discuss an example that falls outside this class.

Let us consider the resolved hypersurface X of degree 8 in $\mathbb{P}^4_{\{2,2,2,1,1\}}$. We refer to [11] for extensive details of this model. The space X can be thought of as a K3-fibration over $\mathbb{P}_1$ or as the resolution of a singular space with a curve of singularities of the form $\mathbb{C}^2/\mathbb{Z}_2$. This model has the same moduli space as that of section 5 except for the following aspects:

- The $\mathbb{P}^2$ phase is renamed a $\mathbb{P}^1$ phase as this is the base of the fibration.
- The LG and orbifold limit points are now at orbifold points locally of the form $\mathbb{C}^2/\mathbb{Z}_2$.
- The Δ_0 component of the discriminant now intersects C_1 at a point of multiplicity two rather than three.

When we compute the monodromies around the discriminant the real difference appears when we consider Δ_1. Let E be the exceptional divisor in X coming from the resolution of the curve of singularities. E is a product of a genus 3 curve Z and $\Gamma \cong \mathbb{P}^1$. We associate Δ_1 with the collapse of E down to Z. The discussion in section 5.2 concerned an exceptional divisor contracting to a *point* and so cannot be applied to monodromy around Δ_1.

The computation of the monodromy on Chern characters was computed in [45]. The result may be rephrased as follows. For a divisor E collapsing to a curve Z of genus g, monodromy around Δ_1 is given by

$$ch(\mathscr{F}) \mapsto ch(\mathscr{F}) - \langle \mathcal{O}_E + (1-g)\mathcal{O}_\Gamma, \mathscr{F}\rangle\, ch(\mathcal{O}_\Gamma) + \langle \mathcal{O}_\Gamma, \mathscr{F}\rangle\, ch(\mathcal{O}_E), \tag{28}$$

where $\Gamma \cong \mathbb{P}^1$ is the inverse image of a point for the blow-down.

The Fourier–Mukai transform given by (6) is incompatible with (28). A more general form which is consistent is given by Horja in [20, 39].[6] We refer to these references for more details.

Given the form of the monodromy (28) it is easy to reproduce all the corresponding results of section 5 for this example.

6.2 A reducible exceptional divisor

One might have got the impression from the localization argument in section 5.2 that we can understand the monodromy associated to an orbifold singularity by studying the little Calabi–Yau W living inside the exceptional divisor E. Indeed this argument shows that

[6] I thank P. Horja for confirming that his construction is consistent with (28).

the analysis we did in section 5 for a Calabi–Yau threefold would also apply locally to a Calabi–Yau fivefold which has a $\mathbb{Z}_{18}$ orbifold singularity given by the action

$$(z_1, z_2, z_3, z_4, z_5) \mapsto (\alpha^9 z_1, \alpha^6 z_2, \alpha z_3, \alpha z_4, \alpha z_5), \tag{29}$$

where $\alpha = \exp(2\pi i/18)$. This is because the resolved $\mathbb{P}^4_{\{9,6,1,1,1\}}$ is the exceptional divisor for this five-dimensional orbifold.

While this is useful in some circumstances it does not mean that any orbifold analysis can be reduced to a Calabi–Yau computation in lower dimensions. The problem is that the exceptional divisor for an orbifold singularity may be *reducible*. Indeed one generically expects an exceptional divisor be to reducible. In this case the notion of the Calabi–Yau "inside" the exceptional divisor makes no sense.

At least in the context of toric cases we can make some general statements about the difference between a reducible and irreducible exceptional divisor. The impression one might have been left with from the above examples is that each particular exceptional divisor E is associated with its own component Δ_E of the discriminant divisor. In this case one then associates monodromy around Δ_E with a process involving the collapse of E.

In the Batyrev [26] way of describing Calabi–Yau n-folds, one has a set of points, $\mathcal{A}$, lying in a hyperplane in $\mathbb{R}^{n+2}$. Vectors from the origin to these points generate the one-dimensional edges of a fan. By the usual algorithm in toric geometry this fan gives the canonical line bundle over some $(n+1)$-dimensional variety V. X is then the "little Calabi–Yau" living inside V.

Let Q by the convex hull of $\mathcal{A}$. One can show (see chapter 10 of [42]) that the irreducible components of the discriminant are determined by faces of Q of various dimensions. For a nontrivial component we require at least $k+2$ points in a k-dimensional face.

If each face of Q has at most one point in its interior then each divisor associated to this interior point gets its own component of the discriminant. This was the case for the examples studied above and was the case for all the examples studied in [23–25]. It is precisely when we have a reducible exceptional divisor that this fails.

We will consider an example of this. For a change we will use a K3 surface rather than a Calabi–Yau threefold. The results generalize easily to the threefold case.

Consider the surface X of degree 12 in $\mathbb{P}^3_{\{4,3,3,2\}}$. This has $\mathbb{C}^2/\mathbb{Z}_2$ singularities at 3 points and $\mathbb{C}^2/\mathbb{Z}_3$ singularities at 4 points. Each $\mathbb{Z}_2$ singularity is resolved by a single $\mathbb{P}^1$ and each $\mathbb{Z}_3$ singularity is resolved by a sum of two $\mathbb{P}^1$'s intersecting at a point. The fact that the K3 is embedded in the given ambient space ties the blow-ups of the points of similar type together. There are in fact only four Kähler degrees of freedom — the overall size of the initial weighted projective space, one blow-up of all the $\mathbb{Z}_2$ fixed points and two blow-ups for all the $\mathbb{Z}_3$ fixed points.

We list the coordinates of the points $\mathcal{A}$ in table 1. We also show the divisor classes in terms of a basis (H, A, B, C) associated to these vectors. These divisor classes restrict to form a partial basis of $H_2(X) = H^2(X)$. We then demand that the Kähler form of X lies

a_0	0	0	0	1	$-2H$
a_1	0	0	1	1	B
a_2	1	0	0	1	C
a_3	1	2	0	1	C
a_4	-3	-3	-2	1	$-H+A$
a_5	-2	-2	-1	1	$2H-2A+B$
a_6	-1	-1	0	1	$A-2B$
a_7	1	1	0	1	$H-2C$

Table 1: The point-set $\mathcal{A}$ for the K3 example.

in the span of these generators. This basis has been chosen so that the resulting slice of the Kähler cone is the positive orthant.

We will only concern ourselves with monodromies associated to the $\mathbb{Z}_3$ blow-ups. To do this we only consider variations in the A and B components of the Kähler form. The two divisor classes of interest are those associated to a_5 and a_6 having classes $2H-2A+B$ and $A-2B$ respectively. These each intersect X in four copies of $\mathbb{P}^1$. Each $\mathbb{P}^1$ from one set intersects one member from the other set of $\mathbb{P}^1$'s in one point. Together these form the resolutions of the four $\mathbb{Z}_3$ fixed points. Note that the set $\{a_4,a_5,a_6,a_1\}$ lies on a straight line.

Let us fix algebraic coordinates

$$x=\frac{a_4a_6}{a_5^2},\quad y=\frac{a_1a_5}{a_6^2}. \tag{30}$$

In terms of these, the component of the discriminant associated with the line $\{a_4,a_5,a_6,a_1\}$ is

$$\Delta_1=27x^2y^2-18xy-1+4x+4y. \tag{31}$$

There are two other components of the discriminant — the primary component and one associated to the three $\mathbb{Z}_2$ fixed points. Δ_1 is the part intrinsically associated to the $\mathbb{C}^2/\mathbb{Z}_3$ resolution.

The discriminant (31) divides the $(-\log x,-\log y)$ into four phases as shown in figure 4. One has the resolved phase, the $\mathbb{Z}_3$ orbifold phase and two phases where one of the pair of $\mathbb{P}^1$'s has been blown up to partially resolve the $\mathbb{Z}_3$ fixed point to something that looks locally like a $\mathbb{Z}_2$ fixed point. We will call the latter two phases $\mathbb{Z}_2^{(A)}$ and $\mathbb{Z}_2^{(B)}$ since they are associated to blowing down using the A or B component of the Kähler form respectively.

Note again that there is only a *single* irreducible component, Δ_1, of the divisor associated to this picture. The same component of the discriminant is responsible for blowing down either of the $\mathbb{P}^1$'s.

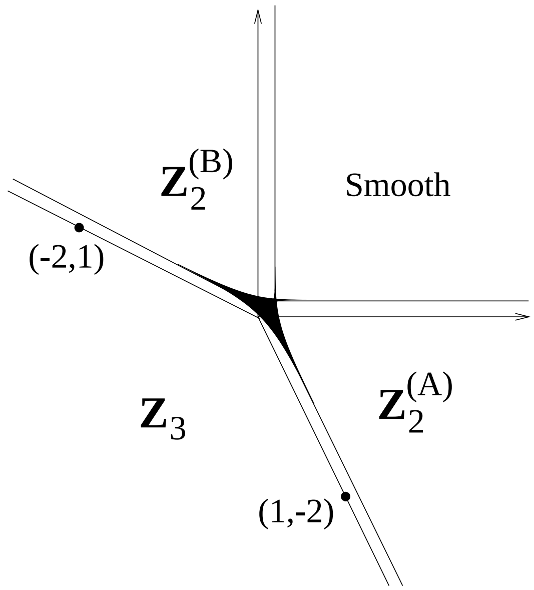

Figure 4: The $\mathbb{C}^2/\mathbb{Z}_3$ resolution phase space.

Now, of course, the moduli space is really four dimensional. We want to think of figure 4 as representing a slice of the moduli space, where the H and C components of the Kähler form have been taken to infinity. Equivalently, think of figure 4 as the toric fan of a two-dimensional subspace of the moduli space associated to this limit.

Let $\mathcal{O}_{a_5}$ be the sum of the four structure sheaves of the $\mathbb{P}^1$'s associated to a_5. Similarly $\mathcal{O}_{a_6}$ is supported only over the four $\mathbb{P}^1$'s associated to a_6. Now consider the $\mathbb{P}^1$ in the moduli space connecting the large radius limit to the $\mathbb{Z}_2^{(A)}$ limit point where the $\mathbb{P}^1$'s associated to a_5 are blown down. Let us denote this by C_A. We know the monodromy around the large radius limit within C_A multiplies the Chern characters by e^A. Since Δ_1 hits C_A transversely we expect that monodromy around the discriminant point is given by $\mathscr{K}^K_{\mathcal{O}_{a_5}}$ which induces

$$ch(\mathscr{F}) \mapsto ch(\mathscr{F}) - \tfrac{1}{4}\langle \mathcal{O}_{a_5}, \mathscr{F}\rangle \mathcal{O}_{a_5}, \tag{32}$$

where the factor $\frac{1}{4}$ appears because $\mathcal{O}_{a_5}$ is associated to *four* irreducible divisors.[7]

The $\mathbb{Z}_2^{(A)}$ limit point is associated to a $\mathbb{Z}_2$ orbifold point and so we expect going twice around this point gives something simple. One can show that applying $\mathscr{K}^K_{\mathcal{O}_{a_5}}\mathscr{K}^B_A$ twice induces multiplication by $\exp(2H + B)$ on the Chern characters. This should be viewed in the same way as section 5.2.

[7]It is not hard to convince yourself that such a factor is necessary to get the monodromies to come out correctly. It would be nice to explain this factor more completely. Presumably this is similar to a asking for a better understanding of (28).

What about the curve C_B which connects the large radius limit to the $\mathbb{Z}_2^{(B)}$ limit point? In this case the discriminant point induces monodromy given by $\mathscr{K}^K_{\mathcal{O}_{a_6}}$ which induces

$$ch(\mathscr{F}) \mapsto ch(\mathscr{F}) - \tfrac{1}{4}\langle \mathcal{O}_{a_6}, \mathscr{F}\rangle \mathcal{O}_{a_6}. \tag{33}$$

One can then show that applying $\mathscr{K}^K_{\mathcal{O}_{a_6}}\mathscr{K}^B_B$ twice induces multiplication by $\exp(A)$ on the Chern characters.

This is all very well except that $\mathscr{K}^K_{\mathcal{O}_{a_5}}$ and $\mathscr{K}^K_{\mathcal{O}_{a_6}}$ are both monodromies around the *same* component of the discriminant. The reason they are different is that if we fix a base point near the large radius limit then the loop around Δ_1 inside C_A is not homotopic to the loop around Δ_1 inside C_B. This can happen because Δ_1 itself is not smooth. It has a cusp at the point $(x, y) = (\frac{1}{3}, \frac{1}{3})$. If we draw an S^3 around this cusp then we obtain a trefoil knot in the intersection. It is well-known that, for a fixed basepoint, loops around different parts of this knot are not homotopic to each other. This allows for the difference between $\mathscr{K}^K_{\mathcal{O}_{a_5}}$ and $\mathscr{K}^K_{\mathcal{O}_{a_6}}$. Note that this is the very same cusp as the one studied by Argyres and Douglas [46].

Finally let us see how to compute the effect of monodromy going three times around the $\mathbb{Z}_3$ limit point. Let C_{A2} be the curve connecting the $\mathbb{Z}_2^{(A)}$ limit point to the $\mathbb{Z}_3$ limit point. The monodromy within C_{A2} around the $\mathbb{Z}_2^{(A)}$ limit point is identical to the monodromy within C_A around the $\mathbb{Z}_2^{(A)}$ limit point for the reasons given in section 5.3. The Δ_1 component of the discriminant intersects C_{A2} transversely. Since this is associated with collapsing the divisor associated to a_6 we will assume that monodromy around this discriminant point is given by $\mathscr{K}^K_{\mathcal{O}_{a_6}}$. *We therefore claim that monodromy with C_A around the $\mathbb{Z}_3$ limit point is given by $\mathscr{K}^K_{\mathcal{O}_{a_6}}\mathscr{K}^K_{\mathcal{O}_{a_5}}\mathscr{K}^B_A$.*

We may check that this cubes to something nice. Indeed the effect on the Chern characters implies that

$$(\mathscr{K}^K_{\mathcal{O}_{a_6}}\mathscr{K}^K_{\mathcal{O}_{a_5}}\mathscr{K}^B_A)^3 = (\mathscr{K}^B_H)^4. \tag{34}$$

One can show that this is consistent with the global geometry of the moduli space and so our monodromies all have the expected properties.

7 Discussion of the 0-Brane

We have given various rules for how to compute the effect of monodromy on the Chern character of a D-brane. In this last section we will discuss the consequences for a 0-brane.

The 0-brane is of particular interest as it is the basic object used in the construction of Bondal and Orlov [7] to build the target space from the derived category. The fact that the 0-brane can transform into something else under monodromy is one reason why the Bondal and Orlov construction is ambiguous for a Calabi–Yau space.

Note again that the following results could have been guessed using the Picard–Fuchs differential equations. In that language the 0-brane often appears as a constant solution

to the differential equations [41]. The derived category provides a much simpler picture however.

For a Calabi–Yau threefold X, let P be an object in $\mathbf{D}(X)$ which corresponds to a 0-brane. This immediately implies that $ch(P) = p$, where $p \in H^6(X)$ is Poincaré dual to a point. Under monodromy about Δ_0 we have

$$\begin{aligned} p \mapsto p &+ \int_X p \wedge td(X) \\ &= p + 1. \end{aligned} \tag{35}$$

That is, the 0-brane always picks up a 6-brane charge upon an orbit around Δ_0.

Now consider the other monodromies in this paper. They all involve taking the structure sheaf $\mathcal{O}_E$ of some collapsing cycle E and computing $\langle \mathcal{O}_E, p \rangle$. If E is of dimension less than 6 then this inner product is always zero. Thus the 0-brane undergoes no monodromy about these kinds of components in the discriminant.

We therefore make the following

Conjecture 3 *The 0-brane undergoes monodromy if and only if we circle the primary component, Δ_0, of the discriminant.*

If we begin in a large radius smooth Calabi–Yau phase, which other phases may we visit without crossing a wall in the phase diagram which contains Δ_0? In other words, over what area of the phase diagram can we fix a choice of 0-brane without worrying about monodromy? The answer consists of the so-called "geometric phases" or "partially enlarged Kähler moduli space" of [28].

The phases correspond to triangulations of Batyrev's reflexive polytope [26, 28]. The statement that a phase is geometric corresponds to every simplex in the triangulation having the unique point in the interior of the polytope as a vertex.

Comparing to the example in section 5 for example, the geometric phases consist of the Calabi–Yau phase and the orbifold phase where we have a three complex dimensional picture of the target space. Indeed, these phases are separated only by Δ_1 around which the 0-brane has no monodromy.

The geometric phases consist of those reached from the Calabi–Yau phase only by blowing down subspaces. If one reduces the overall dimension of the target space then one must cross a wall containing Δ_0. There are also exotic "exoflop" transitions [28] where part of the target space remains three-dimensional but a lower-dimensional part grows out of the side of the target space. These exoflops also involve crossing a Δ_0 boundary and are not considered geometric.

Proving the statement about these phases is an application of the combinatorics discussed in chapter 11 of [42]. There is an object η_T which is a function on the cones of the secondary fan. If η_T changes as you pass to a neighbouring cone then the wall contained Δ_0. It is

easy to show that η_T changes as you pass from a geometric phase to a non-geometric phase. This result is essentially contained in corollary 4.5 of chapter 11 of [42]. It is a more tedious exercise to show that passing between geometric phases of threefolds keeps η_T constant.

This result appears to jibe nicely with the Bondal and Orlov construction. We may consistently tie the derived category to a target space interpretation so long as we confine ourselves to geometric phases. Once we leave these phases then the 0-brane undergoes monodromy and we acquire an ambiguity in the way we construct the target space.

Finally we should note that there can still be an ambiguity in what exactly is called a 0-brane in the geometric phases. If there are two or more smooth phases related by flops then each phase has its own 0-branes. The exact way these branes are related to each other was given by Bridgeland [47].

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References

[1] R. Minasian and G. Moore, *K-Theory and Ramond-Ramond Charge*, J. High Energy Phys. **11** (1997) 002, hep-th/9710230.

[2] E. Witten, *D-branes and K-Theory*, J. High Energy Phys. **12** (1998) 019, hep-th/9810188.

[3] M. Kontsevich, *Homological Algebra of Mirror Symmetry*, in "Proceedings of the International Congress of Mathematicians", pages 120–139, Birkhäuser, 1995, alg-geom/9411018.

[4] M. R. Douglas, *D-Branes, Categories and N=1 Supersymmetry*, hep-th/0011017.

[5] E. Sharpe, *D-branes, Derived Categories, and Grothendieck Groups*, Nucl. Phys. **B561** (1999) 433–450, hep-th/9902116.

[6] A. Kapustin and D. Orlov, *Vertex Algebras, Mirror Symmetry, and D-Branes: The Case of Complex Tori*, hep-th/0010293.

[7] A. Bondal and D. Orlov, *Reconstruction of a Variety from the Derived Category and Groups of Autoequivalences*, alg-geom/9712029.

[8] P. S. Aspinwall and R. Y. Donagi, *The Heterotic String, the Tangent Bundle, and Derived Categories*, Adv. Theor. Math. Phys. **2** (1998) 1041–1074, hep-th/9806094.

[9] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A Pair of Calabi–Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nucl. Phys. **B359** (1991) 21–74.

[10] D. R. Morrison, *Mirror Symmetry and Rational Curves on Quintic Threefolds: A Guide For Mathematicians*, J. Amer. Math. Soc. **6** (1993) 223–247, alg-geom/9202004.

[11] P. Candelas et al., *Mirror Symmetry for Two Parameter Models — I*, Nucl. Phys. **B416** (1994) 481–562.

[12] P. Candelas, A. Font, S. Katz, and D. R. Morrison, *Mirror Symmetry for Two Parameter Models — II*, Nucl. Phys. **B429** (1994) 626–674, hep-th/9403187.

[13] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, *Mirror Symmetry, Mirror Map and Applications to Calabi–Yau Hypersurfaces*, Commun. Math. Phys. **167** (1995) 301–350, hep-th/9308122.

[14] I. Brunner, M. R. Douglas, A. Lawrence, and C. Romelsberger, *D-branes on the Quintic*, JHEP **08** (2000) 015, hep-th/9906200.

[15] D.-E. Diaconescu and J. Gomis, *Fractional Branes and Boundary States in Orbifold Theories*, hep-th/9906242.

[16] D.-E. Diaconescu, M. R. Douglas, and J. Gomis, *Fractional Branes and Wrapped Branes*, JHEP **02** (1998) 013, hep-th/9712230.

[17] D.-E. Diaconescu and C. Romelsberger, *D-branes and Bundles on Elliptic Fibrations*, Nucl. Phys. **B574** (2000) 245–262, hep-th/9910172.

[18] M. Kontsevich, 1996, Rutgers Lecture, unpublished.

[19] P. Seidel and R. P. Thomas, *Braid Groups Actions on Derived Categories of Coherent Sheaves*, math.AG/0001043.

[20] R. P. Horja, *Hypergeometric Functions and Mirror Symmetry in Toric Varieties*, math.AG/9912109.

[21] S. Hosono, *Local Mirror Symmetry and Type IIA Monodromy of Calabi-Yau Manifolds*, hep-th/0007071.

[22] B. Andreas, G. Curio, D. Hernandez Ruiperez, and S.-T. Yau, *Fourier–Mukai Transforms and Mirror Symmetry for D-Branes on Elliptic Calabi–Yau*, math.AG/0012196.

[23] S. Govindarajan and T. Jaramayan, *D-branes, Exceptional Sheaves and Quivers on Calabi-Yau manifolds: From Mukai to McKay*, hep-th/0010196.

[24] A. Tomasiello, *D-branes on Calabi-Yau Manifolds and Helices*, hep-th/0010217.

[25] P. Mayr, *Phases of Supersymmetric D-branes on Kähler Manifolds and the McKay Correspondence*, hep-th/0010223.

[26] V. V. Batyrev, *Dual Polyhedra and Mirror Symmetry for Calabi–Yau Hypersurfaces in Toric Varieties*, J. Alg. Geom. **3** (1994) 493–535.

[27] E. Witten, *Phases of $N = 2$ Theories in Two Dimensions*, Nucl. Phys. **B403** (1993) 159–222, hep-th/9301042.

[28] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, *Calabi–Yau Moduli Space, Mirror Manifolds and Spacetime Topology Change in String Theory*, Nucl. Phys. **B416** (1994) 414–480.

[29] S. I. Gelfand and Y. I. Manin, *Homological Algebra*, Encyclopædia of Mathematical Sciences **38**, Springer, 1994.

[30] R. P. Thomas, *Derived Categories for the Working Mathematician*, math.AG/0001045.

[31] R. P. Thomas, *Mirror Symmetry and Actions of Braid Groups on Derived Categories*, math.AG/0001044.

[32] S. Mukai, *Duality Between $\mathbf{D}(X)$ and $\mathbf{D}(\hat{X})$ with its application to Picard Sheaves*, Nagoya Math. J. **81** (1981) 153–175.

[33] S. Mukai, *Fourier Functor and its Application to the Moduli of Bundles on an Abelian Variety*, Adv. Pure Math. **10** (1987) 515–550.

[34] T. Bridgeland, *Equivalences of Triangulated Categories and Fourier–Mukai Transforms*, Bull. London Math. Soc. **31** (1999) 25–34, math.AG/9809114.

[35] J. A. Harvey and G. Moore, *On the algebras of BPS states*, Commun. Math. Phys. **197** (1998) 489–519, hep-th/9609017.

[36] K. Hori, A. Iqbal, and C. Vafa, *D-branes and Mirror Symmetry*, hep-th/0005247.

[37] E. Scheidegger, *D-branes on some One- and Two-parameter Calabi–Yau Hypersurfaces*, JHEP **04** (2000) 003, hep-th/9912188.

[38] A. L. Gorodentsev, *Exceptional Objects and Mutations in Derived Categories*, in A. N. Rudakov et al., editors, "Helices and Vector Bundles", LMS Lecture Notes **148**, pages 57–74, Cambridge, 1990.

[39] P. Horja, *Derived Category Automorphisms from Mirror Symmetry*, to appear.

[40] D. R. Morrison and M. R. Plesser, *Summing the Instantons: Quantum Cohomology and Mirror Symmetry in Toric Varieties*, Nucl. Phys. **B440** (1995) 279–354, hep-th/9412236.

[41] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, *Measuring Small Distances in $N = 2$ Sigma Models*, Nucl. Phys. **B420** (1994) 184–242, hep-th/9311042.

[42] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinski, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, 1994.

[43] D. R. Morrison, *Geometric Aspects of Mirror Symmetry*, math.AG/0007090.

[44] D. Rolfsen, *Knots and Links*, Math. Lect. Series **7**, Publish or Perish, 1990.

[45] S. Katz, D. R. Morrison, and M. R. Plesser, *Enhanced Gauge Symmetry in Type II String Theory*, Nucl. Phys. **B477** (1996) 105–140, hep-th/9601108.

[46] P. C. Argyres and M. R. Douglas, *New Phenomena in SU(3) Supersymmetric Gauge Theory*, Nucl. Phys. **B448** (1995) 93–126, hep-th/9505062.

[47] T. Bridgeland, *Flops and Derived Categories*, math.AG/0009053.